# AN ALGEBRAIC APPROACH TO LINEAR DIFFERENTIAL OPERATORS

JARVIS KENNEDY

Abstract. This undergraduate research report is an introduction to the theory of monic homogeneous linear differential operators over fields of characteristic 0, studied from an algebraic perspective. Using Differential Algebra and Differential Galois Theory, we show that the differential equation $y' = t - y^2$ has no "elementary" solutions.

## 1. Acknowledgments

I would like to begin by giving thanks and acknowledgment to the Natural Sciences and Engineering Research Council of Canada (NSERC), who supported this research. I would also like to thank Professor Yael Karshon for supervising and advising me throughout the duration of this project.

## 2. Introduction

In this report, we aim to prove that the differential equation $y' = t - y^2$ has no elementary solutions. Of course, this equation can be solved using other methods including power series and Bessel functions. However, these solutions are not classified as "elementary" solutions (which has a purely algebraic description). In order to achieve this goal, we will build up some of the more abstract theory of Differential Algebra and Differential Galois Theory. Most of the theory was taken from Andy Magid's *Lectures on Differential Galois Theory* [1], and the many examples (including the equation $y' = t - y^2$) were taken from John Hubbard's paper *A Frist Look at Differential Algebra* [3].

A few of the results in this report were adapted from Magid's book and re-stated or re-proved in a slightly different way. This was done because we felt that there was a simpler or more natural way to state and prove the results. Specifically, Propositions 5.1 and 5.4, and Theorems 4.4 and 5.6 were modified (see the remarks following each Proposition/Theorem).

This report was written for a reader who has previously studied Galois Theory. For more on Algebraic Groups and Galois Theory, see [4] and [5] respectively.

## 3. Differential Rings and Fields

**Conventions.**

In the context of this report, a ring will always be understood to be a commutative ring with unity.

---

*Date*: August 28, 2018.

3.1. **Definition.** A *differential ring* is a ring $R$ together with a derivation, that is, a map $D_R : R \to R$ satisfying,

- $D_R(a+b) = D_R(a) + D_R(b)$
- $D_R(ab) = aD_R(b) + D_R(a)b$

for every $a, b \in R$.

When the context is clear, the subscript $R$ will be dropped, and often we will use the familiar notation $a'$ for $D(a)$. In the case that $R$ is a field, it will be called a differential field. Throughout, we will always consider fields that have characteristic 0.

From this definition, we get immediate (and expected) properties of any derivation.

- $D(1) = 0$
- $D(x^n) = nx^{n-1}D(x)$ for $n \geq 1$
- $D(x/y) = \frac{yD(x)-xD(y)}{y^2}$ for $y$ a unit.

The sub-ring of *constants* is the kernel of $D$. In the case of fields, this is a subfield.

An *extension* of differential rings, is a ring extension in the usual sense ($S \supset R$), for which $D_S$ restricted to $R$ is $D_R$.

A *homomorphism* of differential rings $\sigma : R \to S$, is a ring homomorphism for which $\sigma(D_R(a)) = D_S(\sigma(a))$.

Now we consider some basic concepts in ring theory; quotients, fields of fractions, and tensor products.

Let $R$ be a differential ring. A *differential ideal* is an ideal $I$ of $R$ which is closed with respect to the derivation. It follows that the quotient $R/I$ is a differential ring with $D_{R/I}(a+I) = D_R(a) + I$. This derivation is well defined, since $D_R(I) \subset I$.

Next we consider rings of fractions. Let $R$ be a differential ring and $Q$ a multiplicatively closed subset of $R$ containing 1 and not 0. Then we define $D : Q^{-1}R \to Q^{-1}R$ by

$$D(a/b) = \frac{bD(a) - aD(b)}{b^2}.$$

This turns $Q^{-1}R$ into a differential ring. With this derivation, the field of fractions of a differential integral domain is a differential field. For more details, see page 2 of [1].

Next, let $R$ be a differential ring, and let $S$ and $T$ be differential $R-$algebras. We can turn the tensor product $S \otimes_R T$ into a differential $R-$algebra by defining

$$D(a \otimes b) = D_S(a) \otimes b + a \otimes D_T(b).$$

**Linear Differential Operators.**

3.2. **Definition.** Let $F$ be a differential field and $\alpha_0, \ldots, \alpha_{n-1} \in F$. A *monic linear homogeneous differential operator of order $n$ over $F$* is a map $L : F \to F$ that has the form

$$L(y) = y^{(n)} + \alpha_{n-1}y^{(n-1)} + \ldots + \alpha_0 y$$

where

$$y^{(m)} = \underbrace{D(D(\ldots D}_{\text{m-times}}(y)\ldots)).$$

Homogeneous linear differential operators over $F$ play the role in Differential Galois Theory that polynomials play in regular Galois Theory. Hence, we wish to study solutions to $L(y) = 0$, and the splitting field of $L$ over $F$ (called Picard-Vessiot extensions, the precise definition is given in section 4). As already mentioned, we require the splitting field, $E$, to extend both $F$ and the derivation $D_F$. This gives rise to some questions. How can we be sure that such an extension $E$ with derivation $D_E$ exists? If one does, is it unique? Is the derivation unique?

In the case of regular Galois Theory, we can take the base field to be $\mathbb{Q}$ and the splitting field to be some subfield of $\mathbb{C}$. Our questions about existence are then answered by the Fundamental Theorem of Algebra. In the differential case, we may take an analogous approach. That is, we may take the base field to be the rational functions $\mathbb{C}(t)$, and the splitting field to be a subfield of the meromorphic functions, $\mathcal{M}(U)$ (quotients of analytic functions defined on an open subset $U$ of $\mathbb{C}$), with the usual derivation.

The reason we can do this is the existence and uniqueness theorem for ordinary differential equations over $\mathbb{C}$: If $U$ is a simply connected open subset of $\mathbb{C}$, and $\alpha_0, \ldots, \alpha_{n-1}$ are analytic on $U$, then the differential equation

$$L(y) = y^{(n)} + \alpha_{n-1}y^{(n-1)} + \ldots + \alpha_0 y = 0$$

has a unique solution in $\mathcal{M}(U)$, for any $t_0$ in $U$ and any initial conditions

$$y(t_0) = y_0,\ y'(t_0) = y_1, \ldots, y^{(n-1)}(t_0) = y_{n-1}$$

(see [2]). As will be shown in the next section, this leads to the fact that there are $n$ solutions linearly independent over constants in $\mathcal{M}(U)$, and the splitting field will be some differential subfield of $\mathcal{M}(U)$.

We can actually say more than this. As we shall see, we can find a unique splitting field for any homogeneous linear differential operator over more general fields. However, we will often come back to the example of meromorphic functions to illustrate results. For now, we define the splitting field in the case of meromorphic functions.

3.3. **Definition.** Let $U$ be a simply connected open subset of $\mathbb{C}$, $F$ be a subfield of $\mathcal{M}(U)$, and $L(y) = y^{(n)} + \alpha_{n-1}y^{(n-1)} + \ldots + \alpha_0 y$ be a differential operator where each $\alpha_i \in F$ is analytic on $U$. The *splitting field* of $L$ over $F$ in $\mathcal{M}(U)$, denoted $E_L$, is the smallest subfield of $\mathcal{M}(U)$ which contains $F$ and every solution to $L(y) = 0$ on $U$.

3.4. *Example.* (see Example 2.6 of [3]) One of the simplest examples is the differential operator $L(y) = y - y'$ over $\mathbb{C}(t)$. The coefficients are analytic on all of $\mathbb{C}$, so the splitting field will be a differential subfield of $\mathcal{M}(\mathbb{C})$. The solutions to this operator are the functions for which $y' = y$, which are functions of the form $Ce^t$, where $C \in \mathbb{C}$. From this, it should be more or less clear that the splitting field is the space of functions whose elements look like

$$\frac{P_0(t) + P_1(t)e^t + \ldots + P_m(t)e^{mt}}{Q_0(t) + Q_1(t)e^t + \ldots + Q_n(t)e^{nt}},$$

where each $P_i$ and $Q_j$ is a complex polynomial, and the denominator is not identically zero.

**Constants.**

3.5. *Example.* (see page 8 of [1]) Consider the rational functions over the complex numbers, $\mathbb{C}(t)$. Let $\mathbb{C}((t))$ be the corresponding field of formal power series. Then under the usual derivation, $\mathbb{C}((t))$ is a differential field, and $\mathbb{C}(t)$ a differential subfield. Let $f$ be the usual exponential series, so that $D(f) = f$. Now, consider the field $F$, obtained by adjoining $f$ to the field of constants $\mathbb{C}$, and the differential operator $y' - y = 0$ over $F$. Suppose we extend $F$ by a new formal solution $g$, so that $D(g) = g$. This is a superfluous addition, since $F$ already contains the solution $f$. The derivation of their ratio is then

$$D(\frac{g}{f}) = \frac{fD(g) - gD(f)}{f^2} = \frac{fg - gf}{f^2} = 0,$$

so by adding this new solution $g$, we have created a new constant, $\frac{g}{f}$.

We wish to avoid adding superfluous solutions and creating new constants. This leads to the next definition.

3.6. **Definition.** A differential field extension $E \supset F$ is said to be a no new constant extension if the kernel of $D_E$ coincides with the kernel of $D_F$. Otherwise, the extension is said to contain new constants.

We will need a condition on our differential field extensions in order to guarantee that they are no new constant extensions. This next theorem, and its corollary provides this condition.

3.7. **Theorem.** *Let $R$ be a differential integral domain, finitely generated over the differential field $F$. Let $E$ denote the quotient field of $R$, and let $C$ denote the field of constants of $F$. Suppose $E$ contains a constant, $d$, which is not in $C$. If $d$ is not algebraic over $C$, then $R$ contains a proper differential ideal.*

*Proof.* See Theorem 1.17 on page 11 of [1]. □

It is the contrapositive of this theorem which will be useful for us. Stated in the corollary below, it gives us a condition to guarantee a no new constant extension, provided that the field of constants of $F$ is algebraically closed. The condition of algebraic closure will not be an issue, since the examples we care about are $\mathbb{C}(t)$ and $\mathcal{M}(u)$, which have field of constants $\mathbb{C}$.

*nnc*

3.8. **Corollary.** *Let $R$ be a differential integral domain, finitely generated over the differential field $F$. Let $E$ be the quotient field of $R$, and let $C$ be the field of constants of $F$. Assume that $R$ contains no proper differential ideals and that $C$ is algebraically closed. Then the field of constants of $E$ coincides with $C$.*

*Proof.* The proof is immediate from the previous theorem. □

In for this Corollary to be effective, we will need to construct differential integral domains with no proper differential ideals. Given a differential ring $R$, Zorn's lemma guarantees the existence of a proper maximal differential ideal, $I$. The quotient $R/I$ will be a differential ring which contains no proper differential ideals. However, it is slightly more involved to show that $R/I$ is an integral domain.

*R/I-int*

3.9. **Proposition.** *Let $R$ be a differential ring, and $I$ a maximal differential ideal such that the quotient $R/I$ is of characteristic* 0*. Then $R/I$ is an integral domain.*

*Proof.* (see Proposition 1.19 of [1]) Let $S = R/I$. Then $S$ contains no proper differential ideals. Suppose that $a$ and $b$ are nonzero elements of $S$, such that $ab = 0$. The first claim is that $D^k(a)b^{k+1} = 0$ for each $k \in \mathbb{N}$. $ab = 0$ implies that $0 = D(ab) = aD(b) + D(a)b$, and multiplication by $b$ gives $0 = D(a)b^2$. Now, we have the formula

$$0 = D^n(ab) = \sum_{k=0}^{n} \binom{n}{k} D^k(a)D^{n-k}(b)$$

If the claim holds for $k = 1, 2, \dots, n-1$, then by multiplication of the above expanded formula by $b^{n+1}$, we see that all but one term multiplies to zero in the sum, and we are left only with the last term, $0 = D^n(a)b^{n+1}$. Hence the first claim follows by induction.

Let $J$ denote the differential ideal generated by $a$, whose elements are of the form $\sum_{k=0}^{n} s_k D^k(a)$, where $s_k \in S$. Suppose that no power of $b$ is zero. The claim implies that every element of $J$ is multiplied to zero by $b^{n+1}$ for the appropriate $n$. Hence, every element of $J$ is a zero divisor, and so it cannot contain 1. On the other hand $a$ is non-zero, so $J$ is a proper differential ideal, which it cannot be, so $b$ is nilpotent. However, since $b$ could have been an arbitrary zero divisor, the same argument implies every zero divisor of $S$ is nilpotent. In particular $a^n = 0$ for some $n$ which we can choose to be minimal. Then $0 = D(a^n) = na^{n-1}D(a)$. Since $S$ is of characteristic zero, $na^{n-1}$ is non-zero. So $D(a)$ is a zero divisor, and it is also nilpotent. We have shown that the derivative of a zero divisor is again a zero divisor. Repeating this shows that every $D^n(a)$ is a zero divisor, hence nilpotent, and hence the ideal they generate, $J$, is a differential ideal consisting of entirely nilpotent elements. Hence $J$ cannot contain 1, and so again $J$ is a proper ideal, which it cannot be. It follows that no such elements $a$ and $b$ exist, so $S$ is an integral domain. □

## 4. The Wronskian

Let $L$ be a monic linear homogeneous differential operator over the differential field $F$. In this section, we aim to construct a differential field extension of $F$ that contains every solution to the operator $L$. We first deal with the abstract definitions and results, before moving into the familiar setting of $\mathbb{C}(t)$ and $\mathcal{M}(U)$.

4.1. **Definition.** Let $y_1, \dots, y_s$ be elements of a differential ring $R$. Then,

$$w(y_1, \dots, y_s) = \begin{vmatrix} {y_1}^{(0)} & {y_2}^{(0)} & {y_3}^{(0)} & \dots & {y_s}^{(0)} \\ {y_1}^{(1)} & {y_2}^{(1)} & {y_3}^{(1)} & \dots & {y_s}^{(1)} \\ {y_1}^{(2)} & {y_2}^{(2)} & {y_3}^{(2)} & \dots & {y_s}^{(2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ {y_1}^{(s-1)} & {y_2}^{(s-1)} & {y_3}^{(s-1)} & \dots & {y_s}^{(s-1)} \end{vmatrix}$$

is called the *Wronskian determinant* (or just Wronskian) of $y_1, \dots, y_s$.

**Linear Independence over Constants.**

Now we move onto proving the main properties of the Wronskian. For more details on the Wronskian, see chapter 2 of [1].

*w(n+1)=0*

4.2. **Proposition.** *Let $R$ be a differential ring, and $y_1, \dots, y_{n+1}$ in $R$ satisfy the differential equation $L(y) = y^{(n)} + \alpha_{n-1}y^{(n-1)} + \dots + \alpha_0 y = 0$, where each $\alpha_i$ is an element in $R$. Then $w(y_1, \dots, y_{n+1}) = 0$*

*Proof.* (see Proposition 2.7 of [1]) By the definition of the Wronskian, we have

$$w(y_1, \dots, y_{n+1}) = \begin{vmatrix} {y_1}^{(0)} & {y_2}^{(0)} & {y_3}^{(0)} & \dots & {y_{n+1}}^{(0)} \\ {y_1}^{(1)} & {y_2}^{(1)} & {y_3}^{(1)} & \dots & {y_{n+1}}^{(1)} \\ {y_1}^{(2)} & {y_2}^{(2)} & {y_3}^{(2)} & \dots & {y_{n+1}}^{(2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ {y_1}^{(n)} & {y_2}^{(n)} & {y_3}^{(n)} & \dots & {y_{n+1}}^{(n)} \end{vmatrix}$$

Since each $y_i$ solves the equation $L(y) = 0$, this last row is a linear combination of the previous rows, and so the determinant is 0. $\square$

This next proposition gives necessary and sufficient conditions for the Wronskian to vanish.

*w=0*

4.3. **Proposition.** *Let $F$ be a differential field with field of constants $C$. Then $y_1, \dots, y_n \in F$ are linearly dependent over $C$ if and only if $w(y_1, \dots, y_n) = 0$.*

*Proof.* (see Proposition 2.8 of [1]) Assume first that the $y_i$ are linearly dependent over $C$. Then there exists $c_i \in C$ such that not all the $c_i$ are zero, and $\sum_{i=1}^{n} c_i y_i = 0$. Then for any $k$, apply the derivation $k-$times to get $\sum_{i=1}^{n} c_i y_i^{(k)} = 0$. This shows that the $c_i$ form a non trivial solution to the system of equations

$$\sum_{i=1}^{n} y_i^{(k)} x_i = 0, \quad 0 \le k \le n-1.$$

The determinant of the coefficient matrix must be 0 then, and this is exactly the Wronskian of $y_1, \dots, y_n$.

Now assume that the Wronskian is 0. By the same reasoning, the above system has a non trivial solution $c_1, \dots, c_n \in F$. In particular, $\sum_{i=1}^{n} c_i y_i = 0$. Without loss of generality, we may assume that $c_1$ is non-zero, and divide by $c_1$, so that $c_1 = 1 \in C$. Again, we have that

$$\sum_{i=1}^{n} c_i y_i^{(k)} = 0.$$

Then for each $0 \leq k \leq n-2$, we can apply $D$ to this equation to get

$$\sum_{i=1}^{n} y_i^{(k+1)} c_i + \sum_{i=1}^{n} y_i^{(k)} D(c_i) = 0.$$

The first sum is 0 by the preceding equation, and in the second sum, $D(c_1) = 0$. This shows $D(c_2), \ldots, D(c_n)$ is a solution to the system

$$\sum_{i=2}^{n} y^{(k)} x_i, \quad 0 \leq k \leq n-2.$$

Again, the determinant of the coefficient matrix of this system is the Wronskian of $y_2, \ldots, y_n$. If for each $i$, $D(c_i) = 0$, then each $c_i \in C$, and the proof is complete. Otherwise, there is a non trivial solution to the system and the Wronskian must be 0. The proof then proceeds by induction, the hypothesis being that $y_2, \ldots, y_n$ are linearly dependent. □

**The Full Universal Solution Algebra.**

If $L(y)$ is a monic linear homogeneous differential operator of order $n$ over $E$, then the set of solutions to $L(y) = 0$ in $E$ is a vector space over the field of constants of $E$. The previous two propositions give a bound on the dimension of this vector space of solutions.

*vsd¡n*

4.4. **Theorem.** *Let $L(y)$ be a monic linear homogeneous differential operator of order $n$ over a differential field $E$, with field of constants $K$, and $V$ the set of solutions to $L(y) = 0$ in $E$. Then $V$ is a vector space over $K$ with dimension at most $n$.*

*Proof.* (see Theorem 2.9 of [1]) The map which sends $y \mapsto L(y)$ on $E$ is a $K-$linear transformation, so its kernel (i.e. $V$) is a $K-$vector space. By 4.2, any $n+1$ elements from $V$ have vanishing Wronskian, and so by 4.3 they are linearly dependent. Hence, $V$ must be finite dimensional of dimension at most $n$. □

4.5. *Remark.* In Theorem 2.9 of [1], Magid states the result for an operator over a field $F$ and an extension $E \supset F$. However, any operator over $F$ is automatically an operator over $E$, so we omit $F$ in the statement of the proposition.

4.4 gives motivation for the next definition.

4.6. **Definition.** Let $L(y)$ be a monic linear homogeneous differential operator of order n over the differential field $E$. We say that $L$ has a *full set of solutions* in $E$ if the vector space of solutions, $V$, has dimension $n$ over the field of constants of $E$. That is, if there are elements $y_1, \ldots, y_n \in V$ whose Wronskian is non-zero.

Now we construct a space where the operator $L$ has a full set of solutions.

4.7. **Definition.** Let $L(y) = y^{(n)} + \alpha_{n-1}y^{(n-1)} + \ldots + \alpha_0 y$ be a monic linear homogeneous differential operator over the differential field $F$. Let

$$S = F[y_{ij} : 0 \leq i \leq n-1, 1 \leq j \leq n][w^{-1}]$$

be the localization of the polynomial ring $R = F[y_{ij}]$ in $n^2$ variables at $w = det(y_{ij})$. Then define the derivation on $R$ to be

$$D_R(y_{ij}) = y_{i+1,j}, \quad i < n-1$$
$$D_R(y_{n-1,j}) = -\sum_{i=0}^{n-1} \alpha_i y_{ij}$$

and extend to $S$. We call $S$ the *full universal solution algebra* for $L(y)$.

If $P$ is any prime differential ideal of $S$, then the fraction field of $S/P$ is a differential field extension of $F$ in which $L(y) = 0$ has $n$ linearly independent solutions (since we localized at $w = det(y_{ij})$). Just as in Example 1.5, it is entirely possible that we have added superfluous solutions, if $F$ already contained a solution to $L(y) = 0$. This is dealt with in the next section. For now we will return to $\mathbb{C}(t)$ and $\mathcal{M}(U)$.

**The Airy Operator.**

First, we will express our differential equation $L(y) = 0$, in the form of a matrix equation $A_L\mathbf{y} = \mathbf{y}'$, where

$$A_L := \left(\begin{array}{c|ccc} 0 & & & \\ \vdots & & I_{n-1} & \\ 0 & & & \\ \hline -\alpha_0 & -\alpha_1 & \ldots & \alpha_{n-1} \end{array}\right), \quad \mathbf{y} = \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix}, \quad \mathbf{y}' = \begin{pmatrix} y' \\ y'' \\ \vdots \\ y^{(n)} \end{pmatrix}$$

If $y_1, \ldots, y_n$ are linearly independent solutions to $L(y) = 0$ in $\mathcal{M}(U)$, then their Wronskian is the determinant of the matrix $W_L$ whose $i^{th}$ column is $\mathbf{y}_i$. Note that in this definition of the Wronskian, it is only defined up to a non-zero complex constant, since it depends on the choice of basis for the space of solutions. In general, it may be hard to find such a basis, however, we are not out of luck. This next proposition gives a way of computing the Wronskian without explicitly knowing the solutions.

*compute-wronskian*

4.8. **Proposition.** *Using the notation above, the Wronskian satisfies the differential equation*

$$y' = Tr(A_L)y$$

*and hence has the form*

$$w(t) = w(t_0) \exp\left[\int_{t_0}^{t} Tr(A_L(s))ds\right]$$

*where $t_0$ is any point in $U$, and the integral is along any path from $t_0$ to $t$. In particular, the Wronskian is always contained in a differential extension obtained by adjoining an exponential of an anti-derivative.*

*Proof.* See Proposition 5.5 of [3]. $\square$

4.9. *Example.* (See Example 5.7 of [3]) Consider the Airy operator, $L_A(y) = y'' - ty$. Then we have

$$A_{L_A} = \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}.$$

We can compute the Wronskian using 4.8. Since $Tr(A_{L_A}) = 0$, we have

$$w_{L_A} = C \exp \int 0 = C$$

for some nonzero $C \in \mathbb{C}$. So the Wronskian is a nonzero constant. We will come back to this example in section 5.

An interesting corollary to 4.8 is that the Wronskian is either never 0, or always 0.

*w-always-0*

4.10. **Corollary.** *Continuing with the notation as above, suppose the Wronskian is zero at some point $t$ in $U$. Then the Wronskian is zero at every $t$ in $U$.*

*Proof.* If $w(t) = 0$, then

$$0 = w(t_0) \exp \left[ \int_{t_0}^{t} Tr(A_L(s))ds \right]$$

$w(t_0)$ must be zero since the exponential function is never 0. However, $t_0$ was arbitrary, so $w(t_0) = 0$ for every $t_0$ in $U$. $\square$

In section 2, we claimed that the existence and uniqueness theorem for differential equations guaranteed that there was a maximal number of linearly independent solutions to $L(y) = 0$ in $\mathcal{M}(U)$. We now prove this assertion.

4.11. **Proposition.** *Suppose $U$ is a simply connected open subset of $\mathbb{C}$, and $\alpha_0, \ldots, \alpha_{n-1}$ are analytic on $U$. If there exists a unique solution to*

$$L(y) = y^{(n)} + \alpha_{n-1}y^{(n-1)} + \ldots + \alpha_0 y = 0$$

*in $\mathcal{M}(U)$ for any $t_0$ in $U$ and any initial conditions*

$$y(t_0) = y_0,\ y'(t_0) = y_1, \ldots, y^{(n-1)}(t_0) = y_{n-1}$$

*then there exists $n$ linearly independent solutions to $L(y) = 0$ in $\mathcal{M}(U)$.*

*Proof.* For each $0 \leq i \leq n-1$, and $0 \leq m \leq n-1$ consider the initial conditions

$$y_i(t_0)^{(m)} = \begin{cases} 1 & \text{if } i = m \\ 0 & \text{otherwise} \end{cases}$$

and apply the existence and uniqueness theorem. Then the matrix

$$\begin{pmatrix} {y_0}^{(0)}(t_0) & {y_1}^{(0)}(t_0) & {y_2}^{(0)}(t_0) & \ldots & {y_{n-1}}^{(0)}(t_0) \\ {y_0}^{(1)}(t_0) & {y_1}^{(1)}(t_0) & {y_2}^{(1)}(t_0) & \ldots & {y_{n-1}}^{(1)}(t_0) \\ {y_0}^{(2)}(t_0) & {y_1}^{(2)}(t_0) & {y_2}^{(2)}(t_0) & \ldots & {y_{n-1}}^{(2)}(t_0) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ {y_0}^{(n-1)}(t_0) & {y_1}^{(n-1)}(t_0) & {y_2}^{(n-1)}(t_0) & \ldots & {y_{n-1}}^{(n-1)}(t_0) \end{pmatrix}$$

is just the identity matrix, which has determinant 1. It's determinant is the Wronskian, $w(t) = w(y_0, \ldots, y_{n-1})(t)$, evaluated at $t = t_0$. Hence by 4.10, $w(t)$ is nonzero everywhere, and by 4.3, $\{y_0, \ldots, y_{(n-1)}\}$ are linearly independent over $\mathbb{C}$. By 4.4, this is a maximally linearly independent set of solutions and hence $\mathcal{M}(U)$ contains a full set of solutions to $L(y) = 0$. $\square$

## 5. Picard-Vessiot Extensions

We saw in Example 2.5 that adding a superfluous solution resulted in a new constant. This first Proposition generalizes that result.

*newconstants*

5.1. **Proposition.** *Let $L$ be a monic linear homogoeneous differential operator of order $n \geq 1$ over the differential field $K$. Let $E \supset K$ be a differential extension. Denote the subfields of constants by $C_E$ and $C_K$ respectively, and the vector spaces of solutions to $L = 0$ by $V_E$ and $V_K$ respectively. Suppose that $V_K$ has dimension $n$ over $C_K$. Then $V_E \supsetneq V_K$ if and only if $C_E \supsetneq C_K$.*

*Proof.* (See Proposition 3.1 of [1]) If $C_E \supsetneq C_K$, then pick $c \in C_E \setminus C_K$, and $v \neq 0 \in V_K$. Then $cv \in E$ is a solution to $L = 0$, however $cv \notin K$. If it were, then $c = cvv^{-1} \in K$, however $D(c) = 0$, so we would have $c \in C_K$.

Now assume that $V_E \supsetneq V_K$. Then we may pick $y_1, \ldots y_n$ in $V_E$, such that

$$w(y_1, \ldots, y_n) \neq 0$$

and $y_1 \notin V_K$. Then consider the differential field generated by the $y_i$, $F = K\langle y_1, \ldots, y_n\rangle$. Then the $y_i$ are in $F$ with nonzero Wronskian, so they form a basis for $V_F$ over $C_F$. However, we can also choose $z_1, \ldots, z_n \in V_K \subset F$ with $w(z_1, \ldots, z_n) \neq 0$ since $K$ contains a full set of solutions. So, the $z_i$ form a basis for $V_F$ over $C_F$ as well as $V_K$ over $C_K$. This gives that the $C_F$ span of the $z_i$ is $V_F$, and hence that $C_F$ properly contains $C_K$. If it did not, then the $C_F$ span of the $z_i$ would be $V_K$, but we know that $y_1$ is not in $V_K$. Therefore, $F$ (and hence $E$), contains a constant not in $K$. $\square$

5.2. *Remark.* In Proposition 3.1 of [1] only one direction is stated and proved. Also, it is stated for a chain of three fields, $E \supset K \supset F$, which follows from our proposition.

With the above proposition in mind, what we wish to do is construct an extension that is generated over $F$ by a full set of solutions to $L = 0$, and contains no new constants. This will be a minimal extension which has a full set of solutions to $L = 0$, and will be the more general definition of the splitting field (3.3). These extensions are called Picard-Vessiot extensions, and are defined below.

5.3. **Definition.** Let $L$ be a monic linear homogeneous differential operator of order $n$ over the differential field $F$. A differential extension field $E \supset F$ is called a *Picard-Vessiot* extension of $F$ for $L$ if:

(1) E is generated over $F$ as a differential field by the set of solutions $V$ of $L = 0$ in $E$. That is $E = F\langle V\rangle$ (the smallest field containing both $F$ and $V$);
(2) E contains a full set of solutions to $L = 0$ (there are $y_i \in V$, $1 \leq i \leq n$, such that $w(y_1, \ldots, y_n) \neq 0$).
(3) E is a no new constant extension of $F$

This definition agrees with our previous definition of the splitting field. Every splitting field in the sense of 3.3 is a Picarrd-Vessiot extension, and every Picard-Vessiot extension of $F \subset \mathcal{M}(U)$ arises as the splitting field for some monic linear homogeneous differential operator with coefficients in $F$ (see Theorem 4.8 of [3]).

**Existence and Uniqueness of Picard-Vessiot Extensions.**

We wish to prove that for a given differential operator $L$, its Picard-Vessiot extension is unique up to isomorphism. We must also show that such an extension exists. Then, we will be justified in calling it "the" Picard-Vessiot extension of $F$ for $L$. This next proposition gets us half way to proving the uniqueness.

*E1=E2*

5.4. **Proposition.** *Let $L$ be a monic linear homogeneous differential operator of order $n$ over the differential field $F$ of characteristic $0$. Let $E_1$, and $E_2$ be Picard-Vessiot extensions of $F$ for $L$. Suppose there is a no new constant extension $E \supset F$, that there exists $F-$differential embeddings $\sigma_1 : E_1 \to E$, and $\sigma_2 : E_2 \to E$. Then $E_1$ and $E_2$ are $F-$differentially isomorphic.*

*Proof.* (See Proposition 3.3 of [1]) Let $V_i$, $i = 1, 2$, and $V$ be the vector space of solutions in $E_i$ and $E$ respectively. Note that each of these extensions has the same field of constants $C$. So the dimension of $V_i$ is $n$ and the dimension of $V$ is at most $n$. For any $y$ in $V_i$, $L(\sigma_i(y)) = \sigma(L(y)) = \sigma(0) = 0$, so that $\sigma_i(V_i) \subset V$. This gives that each vector space coincides, since linear independence in $V_i$ corresponds to linear independence in $\sigma_i(V_i)$ by the injectivity of $\sigma_i$. Since each $E_i = F\langle V_i\rangle$, this gives that

$$\sigma_i(F\langle V_i\rangle) = F\langle \sigma_i(V_i)\rangle) = F\langle V\rangle$$

and hence, their ratio, $\sigma_2^{-1} \circ \sigma_1$ is the desired isomorphism. □

5.5. *Remark.* In Proposition 3.3 of [1], the statement of the proposition is that $E_1$ and $E_2$ have the same image in $E$, however the result which is needed is that they are isomorphic.

The more difficult part of proving the uniqueness of Picard-Vessiot extensions is actually constructing a no new constant extension, which each $E_i$ injects into.

*unique*

5.6. **Theorem.** *Let $L$, $F$, $E_1$, and $E_2$ be as above. Suppose that $F$ has an algebraically closed field of constants. Then there is an $F-$differential isomorphism between $E_1$ and $E_2$.*

*Proof.* (See Theorem 3.5 of [1]) Consider $R = E_1 \otimes_F E_2$. $R$ is finitely generated as an algebra over $E_2$, since $E_1$ is finitely generated over $F$ (by a basis for the vector space of solutions $V_1$). Let $Q$ be a proper maximally differential ideal of $R$. Consider the inverse image of $Q$ in $E_1$

$$I = \{a \in E_1 : a \otimes 1 \in Q\}$$

I is a differential ideal in $E_1$, which has no proper ideals other than the zero ideal, since it is a field. If $I = E_1$, then $1 \otimes 1 \in Q$, contradicting $Q$ being proper. So $I$ is the zero ideal. This gives that the map $\sigma_1 : E_1 \to R/Q$, which sends $a \mapsto a \otimes 1 + Q$ is an embedding. For if $a \otimes 1 + Q = b \otimes 1 + Q$, then $(a - b) \otimes 1 \in Q$, so $a - b \in I$, and hence $a = b$. The same is true for the map $\sigma_2 : E_2 \to R/Q$, where $b \mapsto 1 \otimes b$. Now, $R/Q$ has characteristic 0, since if $n(1 \otimes 1) + Q = 0$, then $n \otimes 1 + Q = 0$, so $n \in I$, implying $n = 0$. Hence, by 3.9, $R/Q$ is an integral domain. Now, $R/Q$ is an integral domain, finitely generated over $E_2$ (which has an algebraically closed field of constants), and has no proper differential ideals. So by 3.8, the constants of $R/Q$ coincide with the constants of $E_2$, and hence with $F$. These embeddings extend to embeddings into the fraction field of $R/Q$, E. Since the tensor product is over $F$, $f \otimes 1 = 1 \otimes f$, so we may view $F \subset E$ as

$$F = \left\{ \frac{f \otimes 1 + Q}{1 \otimes 1 + Q} : f \in F \right\},$$

and both embeddings are the identity on $F$. These are differential embeddings. Hence, we have two $F-$differential embeddings of two Picard-Vessiot extensions into a no new constant extension of $F$, so by 5.4, $E_1$ and $E_2$ are $F-$differentially isomorphic. □

5.7. *Remark.* In Theorem 3.5 of [1], Magid first constructs a Picard-Vessiot extension, and then proves that any other Picard-Vessiot extension is isomorphic to it. We instead show that any two abstract Picard-Vessiot extensions are isomorphic, and then construct one explicitly. This made the proof easier to follow. Details were also added to the proof which Magid left out.

So far, we have shown that for a monic homogeneous differential operator $L$ over $F$ which has an algebraically closed field of constants, any two Picard-Vessiot extensions of $F$ for $L$ are isomorphic. However, we still need to show that the Picard-Vessiot extension actually exists.

*existence*

5.8. **Theorem.** *Let $F$ and $L$ be as above. Let $C$ be the field of constants of $F$, $S$ the full universal solution algebra for $L$, and $P$ a proper maximal differential ideal of $S$. Then $P$ is prime and the fraction field, $E$, of $S/P$ is a Picard-Vessiot extension of $F$ for $L$.*

*Proof.* (See Theorem 3.4 of [1]) Since $S$ is finitely generated over $F$ by the solutions to $L = 0$ and the inverse Wronskian, so is $S/P$. By 3.9, $S/P$ is an integral domain, and since $P$ is a maximally differential ideal, $S/P$ contains no proper differential ideals. Again, then, 3.9 applies ($S/P$ has characteristic 0 since $F$ does and $P$ is proper), and the constants of $E$ coincide with $C$. Since $S/P$ is generated by the solutions to $L = 0$, so is $E$, and since the Wronskian is a unit in $S/P$, it is also a unit in $E$, and hence is non-zero. So $E$ contains a full set of solutions to $L = 0$. So, $E$ is generated by the solutions to $L = 0$, contains a full set of solutions, and contains no new constants, hence it is a Picard-Vessiot extension as claimed. □

**Example of a Picard-Vessiot Extension.**

By 5.6, any Picard-Vessiot extension must be isomorphic to the extension constructed in 5.8. However, we already saw an example of a Picard-Vessiot extension in Example 2.4, so a natural question is how this construction is related to Example 2.4.

5.9. *Example.* Consider $L(y) = y - y'$ over $\mathbb{C}(t)$. In this case, the full universal solution algebra is

$$\mathbb{C}(t)[y][w^{-1}]$$

with $y' = y$, and where the Wronskian $w$ is the determinant of the $1 \times 1$ matrix $[y]$, which is just $y$. Hence, it is the localized polynomial ring

$$\mathbb{C}(t)[y][y^{-1}]$$

Consider the natural inclusion map, $i\colon \mathbb{C}(t)[y] \to \mathbb{C}(t)[y][y^{-1}]$, given by $p(y) \mapsto \frac{p(y)}{1}$. We can relate the differential ideals of the localized polynomial ring to the differential ideals of the regular polynomial ring. Let $J$ be any differential ideal of the localized polynomial ring, and consider its inverse image under the inclusion map

$$i^{-1}(J) = \left\{ p(y) \in \mathbb{C}(t) : \frac{p(y)}{1} \in J \right\}$$

It can easily be checked that $i^{-1}(J)$ is a differential ideal of $\mathbb{C}(t)[y]$. Hence, it is generated by a single element, $f(y)$ (since every polynomial ring is a principle ideal domain). It must be the case then that the differential ideal $\left\langle \frac{f(y)}{1} \right\rangle$ is contained in $J$. On the other hand, any element in $J$ has the form $\frac{p(y)}{y^n}$, and so

$$y^n \frac{p(y)}{y^n} = \frac{p(y)}{1}$$

is also in $J$, and hence $p(y)$ is in $i^{-1}(J)$. This gives us that $p(y) = f(y)g(y)$ for some polynomial $g(y)$. Hence, we have

$$\frac{p(y)}{y^n} = \frac{f(y)g(y)}{y^n} = \frac{g(y)}{y^n}\frac{f(y)}{1}$$

and so $J$ is contained in $\left\langle \frac{f(y)}{1} \right\rangle$. This shows that any differential ideal in the localized polynomial ring is generated by the same element which generates its pre-image in the regular polynomial ring. Therefore, we can look for the proper differential ideals in the regular polynomial ring, and their images will give us the corresponding differential ideals in the localized polynomial ring.

Suppose then that $I = \langle f(y) \rangle$ is a differential ideal of $\mathbb{C}(t)[y]$, where

$$f(y) = y^n + \sum_{i=0}^{n-1} a_i y^i$$

then $f(y)'$ must be a multiple of $f(y)$, but

$$f(y)' = ny^n + \sum_{i=0}^{n-1} (a_i' + ia_i) y^i$$

so we must have that

$$f(y)' = nf(y).$$

For any $i \neq 0$, we have the following equation on the ratios of the coefficients

$$n = \frac{a_i' + ia_i}{a_i}$$

and so

$$(n-i)a_i = a_i'.$$

The non zero solutions to these differential equations are exponentials, which are not contained in $\mathbb{C}(t)$, and hence every coefficient must be zero. The case of $i = 0$ gives the same conclusion. This shows that any differential ideal of the polynomial ring is in the form of $\langle y^n \rangle$. The image of this ideal in the localized polynomial ring contains 1, and is hence the entire ring. From this, we conclude that the only differential ideal of the localized polynomial ring is the zero ideal. Since it is the only differential ideal, it is a maximally differential ideal, so the quotient

$$\mathbb{C}(t)[y][y^{-1}]/\langle 0\rangle \cong \mathbb{C}(t)[y][y^{-1}]$$

and the field of fractions is just

$$\mathbb{C}(t)(y)$$

which is isomorphic to the splitting field of Example 2.4 by the map $y \mapsto e^t$, and the identity on $\mathbb{C}(t)$

This example highlights the importance of localization. If we did not localize, then the ideal $\langle y\rangle$ would be a maximally differential ideal, and the quotient would be isomorphic to $\mathbb{C}(t)$, which is not isomorphic to the splitting field given in Example 2.4. Essentially, localization by $y$ is formally declaring $y$ to be nonzero, whereas the quotient $\mathbb{C}(t)[y]/\langle y\rangle$ is formally declaring $y$ to be zero. Since we want our field to contain a full set of solutions, we localize by the Wronskian to ensure that it is nonzero in the field of fractions.

## 6. Differential Galois Theory

Now that we have a differential version of the splitting field, we can consider the group of $F-$automorphisms acting on it. That is, the group of automorphisms which commute with the derivation and restrict to the identity on $F$.

### Differential Galois Groups.

6.1. **Definition.** Let $F$ be a differential field, and $E$ a differential extension of $F$. The *differential Galois Group*, $DGal(E/F)$, is the group of automorphisms $\sigma : E \to E$ which restrict to the identity on $F$, and satisfy $\sigma(D(a)) = D(\sigma(a))$ for every $a \in E$. The group operation is composition of automorphisms.

If $L$ is a monic linear homogeneous differential operator of order $n$ over $F$, and $E \supset F$ its Picard-Vessiot extension, then just as expected $DGal(E/F)$ permutes the solutions of $L(y) = 0$. Since $E$ is generated by the $n-$dimensional space of solutions $V$, the elements of $DGal(E/F)$ are determined by their action on $V$. Thus, we may view $DGal(E/F)$ as a subgroup of $GL(V)$, the group of invertible linear operators on $V$. Once a basis is chosen for $V$, this is naturally isomorphic to the group of $n \times n$ invertible matrices, and this identification will always be made. In fact, it is not just any subgroup, but an algebraic subgroup (See Theorem 4.3 of [3]).

Let us see a few examples of what this means.

6.2. *Example.* (See section 4 of [3]) The additive group $\mathbb{C}$ has lots of subgroups, for instance isomorphic to $\mathbb{Z}$, $\mathbb{Z} \oplus \mathbb{Z}$, $\mathbb{R}$, and many more. However, none of these groups are determined by a finite number of polynomials, since each polynomial has a finite number of roots. They are the zero sets of certain other functions. $\mathbb{Z}$ is determined by $\sin(\pi z)$, and $\mathbb{R}$ is determined by $f(z) = z - \bar{z}$, both of which are not polynomials. The multiplicative group $\mathbb{C}^*$ also has many subgroups, however the only algebraic ones are the multiplicative subgroups consisting of the $n$th roots of unity (determined by $z^n - 1 = 0$).

*G/G0-finite*

6.3. **Theorem.** *Let $G$ be an algebraic group. The connected component containing the identity, $G_0$, is a normal algebraic subgroup of finite index (that is, the quotient $G/G_0$ is finite).*

*Proof.* See section 7.3 of [4]. □

Another characterization of the Picard-Vessiot is one that is taken as a definition in [3]. Here, we will state it as a proposition without proof.

*fixfield=F*

6.4. **Proposition.** *Let $E$ be a Picard-Vessiot extension of $F$ for the monic linear homogeneous differential operator $L$. The set of elements which is fixed by every automorphism $\sigma \in DGal(E/F)$ is precisely $F$.* □

This means that if we can show an element $a$ of $E$ is fixed by every $\sigma$ in $DGal(E/F)$, then we will have shown $a$ is contained in $F$.

*Dgalfinite=algebraic*

6.5. **Proposition.** *Let $L$ be a monic linear differential operator over $F$, and $E$ its Picard-Vessiot extension. Suppose that $DGal(E/F)$ is finite. Then every element of $E$ is algebraic over $F$.*

*Proof.* (See Proposition 4.6 of [3]) Let $v \in E$. Define

$$f := \prod_{\sigma \in DGal(E/F)} (x - \sigma(v))$$

This polynomial is of finite degree and its coefficients are fixed by $DGal(E/F)$, hence by 6.4 they are in $F$. One of the $\sigma$ in the product is the identity, so $f$ has $v$ as a root, and hence $v$ is algebraic over $F$. □

6.5 will be particularly useful when we are in the case of $\mathcal{M}(U)$ and $\mathbb{C}(t)$, since $v$ being algebraic over $\mathbb{C}(t)$ implies that $v$ has finitely many poles.

*FTDGT*

6.6. **Theorem.** *Let $F$ be a differential field with algebraically closed field of constants, and $K \supset F$ a Picard-Vessiot extension. If $M$ is a differential subfield of $K$ such that $F \subset M \subset K$, then $DGal(K/M)$ is an algebraic subgroup of $DGal(K/F)$. Furthermore, this defines an inclusion-reversing bijection between differential subfields of $K$ containing $F$, and the algebraic subgroups of $DGal(K/F)$. Under this bijection, the normal subgroups of $DGal(K/F)$ correspond to the differential subfields $M \subset K$ such that $M \supset F$ is also a Picard-Vessiot extension. In this case, $DGal(M/F) \cong DGal(K/F)/DGal(K/M)$*

*Proof.* See Theorem 6.5 of [1] □

Suppose that $L$ is a monic linear homogeneous differential operator over the differential field $F$ with algebraically closed field of constants, and $E$ its Picard-Vessiot extension. Let $y_1, \ldots, y_n$ be a full set of solutions in $E$, $w$ be their Wronskian, and $V$ be their span over $\mathbb{C}$. If $w$ is not in $F$, then we can create an intermediate Picard-Vessiot extension by adjoining the Wronskian, that is $F \subset F(w) \subset E$. This next proposition describes the structure of $DGal(E/F(w))$.

*Dgal-in-sl2*

6.7. **Proposition.** *Let $L$, $F$, $E$, $w$, and $V$ be as above. Then after identifying $DGal(E/F)$ with a subgroup of $GL(V)$, we have*

$$DGal(E/F(w)) = DGal(E/F) \cap SL(V)$$

*where $SL(V) \subset GL(V)$ is the subgroup of automorphisms of determinant 1.*

*Proof.* See Proposition 4.23 of [1] □

As we saw in Example 3.8, the Wronskian for the Airy operator was some complex constant $C \in \mathbb{C}$. In particular, for the intermediate extension $\mathbb{C}(t)(w)$, we have $\mathbb{C}(t)(w) = \mathbb{C}(t)$, and so by 6.7, the Differential Galois Group $DGal(E_{L_A}/\mathbb{C}(t))$ is a subgroup of $SL_2(\mathbb{C})$. Later, we will determine exactly which subgroup it is.

## 7. Liouvillian Extensions

The existence and uniqueness theorem of section 2 guarantees that solutions exist in $\mathcal{M}(U)$ for differential equations of a certain form, but we wish to know more about these solutions than just their existence. It would be nice if we could express the solutions in terms of the familiar elementary functions from calculus. In this section, we give a condition for when this is possible. Of course, what we mean by elementary function will need to be made precise.

7.1. **Definition.** Let $K$ be a differential field, and $a, b \in K$. $g$ is called an *exponential* of $a$ if $g$ satisfies the differential equation

$$\frac{g'}{a'} = g$$

7.2. **Definition.** An extension $K$ of a differential field $F$ is called a *Liouvillian* extension if there exists a sequence

$$F = F_0 \subset F_1 \subset \ldots \subset F_m = K$$

where each $F_{J+1}$ is either finite algebraic over $F_j$, generated by an antiderivative of an element in $F_j$, or generated by an exponential of an antiderivative of an element in $F_j$.

When the base field $F$ is the rational functions $\mathbb{C}(t)$, we say that the elements of $K$ are elementary functions. Indeed, any familiar elementary function from calculus is contained in a Liouvillian extension. This next proposition gives the condition for an extension to be Liouvillian.

*Liouvillian*

7.3. **Proposition.** *Let $L$ be a linear differential operator over the differential field $F$, and $E$ its Picard-Vessiot extension. Let $G = DGal(E/F)$. The differential field $E$ is contained in a Liouvillian extension if and only if the connected component containing the identity, $G_0$, is a solvable group.*

*Proof.* See sections 25-27 of [6] □

## 8. The equation $y' = t - y^2$

We have finally built up enough theory to show that the equation $y' = t - y^2$ has no elementary solutions. This section will be dedicated to proving this fact. This first result we need is to classify the connected subgroups of $SL_2(\mathbb{C})$.

*subgroups*

8.1. **Proposition.** *Every proper connected subgroup of $SL_2(\mathbb{C})$ is conjugate to one of the following:*

(1) $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}$
(2) $\left\{\begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix} : a \in \mathbb{C}^*\right\}$
(3) $\left\{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{C}\right\}$
(4) $\left\{\begin{pmatrix} a & b \\ 0 & \frac{1}{a} \end{pmatrix} : a \in \mathbb{C}^*, b \in \mathbb{C}\right\}$

*Proof.* The connected subgroups of $SL_2(\mathbb{C})$ are in bijection with the Lie sub-algebras of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ by the exponential map, $exp\colon \mathfrak{sl}_2(\mathbb{C}) \to SL_2(\mathbb{C})$, where

$$exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

So we can find the sub-algebras (up to conjugation) of $\mathfrak{sl}_2(\mathbb{C})$, and exponentiate them to get the connected subgroups of $SL_2(\mathbb{C})$. Since $\mathfrak{sl}_2(\mathbb{C})$ is a 3 dimensional vector space, this is an easy task.

The only 0 dimensional sub algebra is

$$\left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\}$$

which exponentiates to

$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}.$$

Every matrix is conjugate to its Jordan form, so the one dimensional sub-algebras are determined by the possible Jordan forms (diagonalizable versus non-diagonalizable). They are,

(1) $\left\{\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} : a \in \mathbb{C}\right\}$
(2) $\left\{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in \mathbb{C}\right\}$

The first exponentiates to

$$\sum_{n=0}^{\infty}\frac{\begin{pmatrix}a & 0\\ 0 & -a\end{pmatrix}^n}{n!}=\begin{pmatrix}\sum\limits_{n=0}^{\infty}\frac{a^n}{n!} & 0\\ 0 & \sum\limits_{n=0}^{\infty}\frac{(-a)^n}{n!}\end{pmatrix}=\begin{pmatrix}e^a & 0\\ 0 & e^{-a}\end{pmatrix}$$

which gives

$$\left\{\begin{pmatrix}a & 0\\ 0 & \frac{1}{a}\end{pmatrix}: a\in\mathbb{C}^*\right\},$$

since $e^x$ maps surjectively onto $\mathbb{C}^*$. The second exponentiates to

$$\sum_{n=0}^{\infty}\frac{\begin{pmatrix}0 & b\\ 0 & 0\end{pmatrix}^n}{n!}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\begin{pmatrix}0 & b\\ 0 & 0\end{pmatrix}+\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}+\ldots=\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}.$$

The only 2 dimensional sub algebra is

$$\left\{\begin{pmatrix}a & b\\ 0 & -a\end{pmatrix}\right\}.$$

This is because any 2 dimensional subalgebra of $\mathfrak{sl}_2(\mathbb{C})$ is automatically a maximally solvable (Borel) subalgebra (since any two dimensional Lie Algebra is solvable and $\mathfrak{sl}_2(\mathbb{C})$ is not), and Borel subalgebras are conjugate. For $n=2k$, we have

$$\begin{pmatrix}a & b\\ 0 & -a\end{pmatrix}^n=\begin{pmatrix}a^n & 0\\ 0 & (-a)^n\end{pmatrix}$$

and for $n=2k+1$, we have

$$\begin{pmatrix}a & b\\ 0 & -a\end{pmatrix}^n=\begin{pmatrix}a^n & a^{n-1}b\\ 0 & (-a)^n\end{pmatrix}.$$

This 2 dimensional sub-algebra exponentiates to

$$\sum_{n=0}^{\infty}\frac{\begin{pmatrix}a & b\\ 0 & -a\end{pmatrix}^n}{n!}=\sum_{k=0}^{\infty}\frac{\begin{pmatrix}a^{2k} & 0\\ 0 & (-a)^{2k}\end{pmatrix}}{(2k)!}+\frac{\begin{pmatrix}a^{2k+1} & a^{2k}b\\ 0 & (-a)^{2k+1}\end{pmatrix}}{(2k+1)!}$$

for $a\neq 0$ this converges to

$$\begin{pmatrix}e^a & b\frac{sinh(a)}{a}\\ 0 & e^{-a}\end{pmatrix}$$

and this is equivalent to

$$\left\{\begin{pmatrix}a & b\\ 0 & \frac{1}{a}\end{pmatrix}: a\in\mathbb{C}^*, b\in\mathbb{C}\right\}.$$

This concludes the list, since $\mathfrak{sl}_2(\mathbb{C})$ is 3 dimensional. □

All of the theory we have developed has been about monic linear homogeneous differential operators, however the operator we are concerned with in this section is not linear. We must then relate it to one which is linear and we can apply the theory to. Consider again the Airy operator over $\mathbb{C}(t)$,

$$L_A(y)=y''-ty.$$

Since the function $t$ has no poles, the splitting field $E_{L_A}$ is some subfield of $\mathcal{M}(\mathbb{C})$. Suppose that $v$ is a solution to $L_A$, that is $v'' - tv = 0$, and consider the logarithmic derivative $w = \frac{v'}{v}$. Differentiating $w$, we get

$$\begin{aligned} w' &= \frac{vv'' - (v')^2}{v^2} \\ &= \frac{tv^2 - (V')^2}{v^2}, \quad \text{since } L_A(v) = 0 \\ &= t - w^2. \end{aligned}$$

So solutions to $L_A$ produce solutions to $y' = t - y^2$. We can also reverse this process. If $w$ is a solution to $y' = t - y^2$, then $v = \exp(\int w)$ is a solution to $L_A$. Before we proceed any further we must look more closely at first and second order linear operators.

8.2. *Example.* Let $L$ be a first order linear operator, and consider the non-homogeneous equation $L(y) = \beta$. We take the equation $y' + \alpha y = \beta$, multiply by the integrating factor, $w := \exp(\int \alpha)$. Note that $w$ is contained in a Liouvillian extension. This transforms our equation into $(vw)' = w\beta$, which gives the solutions

$$v = \frac{1}{w}\int w\beta.$$

In particular $v$ is contained in a Liouvillian extension.

Now consider a second order monic linear homogeneous operator $L$. Let $v$ be a solution to $L(y) = 0$, and $K = F(v)$. We can show that $E_L \supset K$ is a Liouvillian extension, and hence if $K$ is Liouvillian over $F$, that $E_L \supset F$ is Liouvillian. Let $w$ be another solution linearly independent to $v$. Then we have the Wronskian, $W_L = vw' - wv'$, and hence $w$ satisfies the first order non-homogeneous equation

$$w' - \frac{v'}{v}w = \frac{W_L}{v}$$

over $K$. Here, $\frac{W_L}{v}$ is contained in a Liouvillian extension of $K$. From what we saw above, $E_L \supset K$ is a Liouvillian extension, and hence if there exists a solution to a second order operator contained in a Liouvillian extension, then all solutions are contained in a Liouvillian extension.

We now need to show that no nonzero solution to $L_A$ belongs to a Liouvillian extension. Once this is done, we can conclude that no solution to $y' = t - y^2$ belongs to a Liouvillian extension and we will have completed our goal. As we saw in section 5, $DGal(E_{L_A}/\mathbb{C}(t))$ is a subgroup of $SL_2(\mathbb{C})$. We now will show exactly which subgroup it is.

8.3. **Theorem.** $G = DGal(E_L/\mathbb{C}(t)) = SL_2(\mathbb{C})$

*Proof.* (See Theorem 8.1 of [3]) Let $G_0$ be the connected component containing the identity of $G$. For each of the connected subgroups from 8.1, $(1, 0)^\dagger$ is a common eigenvector. For a contradiction, assume $G_0$ is proper. Then it is conjugate to one of the previous groups and hence each element of $G_0$ would have a common eigenvector

$v$ in $E_{L_A}$. Then for any $\sigma$ in $G_0$ with eigenvalue $a$ for $v$, we would have

$$\sigma(\frac{v'}{v}) = \frac{\sigma(v)'}{\sigma(v)} = \frac{(av)'}{av} = \frac{v'}{v}$$

so $w = \frac{v'}{v}$ is left fixed by $G_0$. Hence, the Picard-Vessiot extension, $M$, which is generated by $w$ is a differential subfield of $E_{L_A}$, and $G_0 \subset DGal(E_{L_A}/M)$ since $G_0$ fixes $w$. Applying the Fundamental Theorem of Differential Galois Theory, we have

$$DGal(M/\mathbb{C}(t)) \cong G/DGal(E_{L_A}/M)$$

Which is a quotient of $G$ by a group containing $G_0$. Hence it is finite, since $G/G_0$ is finite by 6.3. We can then apply 6.5, and see that $w$ is algebraic and so it has finitely many poles.

However, as we saw previously, $w$ satisfies the differential equation $w' = t - w^2$. The goal then is to show that solutions to this differential equation have infinitely many poles, and we will have reached our contradiction.

First, we note that if $w$ is such a solution with finitely many poles, then when we restrict $w$ to the real line, it must be defined on some unbounded interval (since the only points which $w$ is not defined on are its poles). Thus, it is enough to show that the real part of a solution $w$ is only defined on a bounded open interval of the real line.

If $w$ is a solution to the previous differential equation, then $-w$ is a solution to the equation $w' = w^2 - t$, so we may focus our attention here. Restricting $t$ and $w$ to be real, for any $t < -1$, we have

$$w' = w^2 - t > w^2 + 1$$

and so any solution to $w' = w^2 - t$ must climb faster than the solutions to $w' = w^2 + 1$, which are the functions $\tan(t + C)$. Suppose then, that $w$ is real a solution defined on some open interval to the left of $-1$, containing a point $t_0 < -1 - \frac{\pi}{2}$ with $w(t_0) = 0$. For $t > t_0$, $w$ must be above $\tan(t - t_0)$ since the functions agree at $t = t_0$ and $w$ climbs faster. On the other hand, for any $t < t_0$ the only way for $w$ to meet $\tan(t - t_0)$ at $t = t_0$ is if it were below $\tan(t - t_0)$ (since $w$ still climbs faster). In fact if $w(t_0) = s_0$ for some $s_0 \in \mathbb{R}$, the same argument could be made by a shift of the tangent function. This shows that the interval which $w$ is defined on, is of length at most $\pi$. However, as mentioned above, this contradicts $w$ having finitely many poles. Hence, $w$ cannot be algebraic, and our assumption that $G_0 \neq SL_2(\mathbb{C})$ is false. $\square$

8.4. **Corollary.** *No nonzero solution to the Airy operator belongs to a Liouvillian extension of* $\mathbb{C}(t)$*.*

*Proof.* (See Corollary 8.2 of [3]) By 7.3, if a solution (and hence all by Example 7.2) were contained in a Liouvillian extension, then the connected component containing the identity, $G_0 = SL_2(\mathbb{C})$, would be solvable. Solvability is preserved by quotients, so $PSL_2(\mathbb{C}) = SL_2(\mathbb{C})/\{\pm I\}$ would be solvable. However, $PSL_2(\mathbb{C})$ is simple, so it is solvable if and only if it is abelian. This is not the case, so there are no solutions contained in a Liouvillian extension. □

8.5. **Corollary.** *The differential equation* $y' = t - y^2$ *has no solutions which belong to a Liouvillian extension of* $\mathbb{C}(t)$

*Proof.* (See Corollary 8.3 of [3]) Suppose $v$ is such a solution, then $\exp(\int v)$ is contained in a Liouvillian extension of $\mathbb{C}(t)$, and it satisfies the Airy operator, contradicting the previous corollary. □

## References

[1] Magid, A. (1994). Lectures on Differential Galois Theory. University Lecture Series.
[2] Hille, E. (1976). Ordinary differential equations in the complex domain. New York: Wiley.
[3] John H. Hubbard, and Benjamin E. Lundell. A First Look at Differential Algebra. The American Mathematical Monthly, vol. 118, no. 3, 2011, pp. 245261. JSTOR, JSTOR, www.jstor.org/stable/10.4169/amer.math.monthly.118.03.245.
[4] Humphreys, J. E. (1975). Linear algebraic groups. New York: Springer-Verlag.
[5] Dummit, D. S., Foote, R. M. (2004). Abstract algebra.
[6] Kolchin, E. (1948). Algebraic Matric Groups and the Picard-Vessiot Theory of Homogeneous Linear Ordinary Differential Equations. Annals of Mathematics, 49(1), second series, 1-42. doi:10.2307/1969111

*Email address*: `jarvis.kennedy@mail.utoronto.ca`